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C-, PT- and CPT-invariance of pseudo-Hermitian Hamiltonians

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Abstract
We propose a construction of unique and definite metric, η_+, time-reversal operator, T, and an inner-product of eigenstates such that the pseudo-Hermitian matrix Hamiltonians are C-, PT- and CPT-invariant and PT-norm (CPT-norm) is indefinite (definite). Here, P and C denote the generalized parity and charge-conjugation symmetries, respectively. The limitations of the other current approaches are also indicated.

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1. Introduction

The remarkable developments [1–7] of real discrete spectrum of complex PT (P: parity and T: time-reversal) symmetric Hamiltonians have eventually culminated in the revival [10–15] of the concept of pseudo-Hermiticity of a Hamiltonian. The concept of pseudo-Hermiticity was developed in 1950s–60s [9] following a definition of distorted inner-product: $\langle\Psi|\eta\Psi\rangle$ [8], η is called a metric. A Hamiltonian is called pseudo-Hermitian if

$$\eta H \eta^{-1} = H^{\dagger}. \tag{1}$$

Pseudo-Hermiticity is a more general condition than that of PT-symmetry on a Hamiltonian for possessing real eigenvalues [11]. Nevertheless, PT-symmetry is more important in making contact with physical situations and systems. This motivates one to recast pseudo-Hermiticity of a Hamiltonian in terms of PT-symmetry, in some generalized sense.

The most interesting feature of the eigenstates of such Hamiltonians is the indefiniteness [4–9] (positivity–negativity) of the norm which is a consequence of the η-inner-product [9]

$$\langle\Psi_m|\eta\Psi_n\rangle = \epsilon_n \delta_{m,n} \tag{2}$$

where ϵ_n ($=\pm1$) is indefinite (positive–negative). Recall that the usual norm, $\langle\Psi_n|\Psi_n\rangle$, in Hermitian quantum mechanics is positive definite as it represents quantum mechanical probability. Therefore, the construction of a definite metric η_+ so that the norm (2) becomes positive is of fundamental importance. This question is addressed in the present work.

Interestingly, the indefiniteness of the *PT*-norm has given rise to a new direction of investigations. It has been proposed [16] that the indefiniteness of the *PT*-norm indicates the presence of a hidden symmetry called *C* which mimics the charge-conjugation symmetry ($\mathcal{C}$) [17] of the relativistic Dirac field. It has been claimed that the *CPT*-norm will be positive definite.

The next related development [18] caters to the construction of generalized involutary operators *C*, *P*, *T* from the bi-orthonormal [9] basis (Ψ, Φ) of the pseudo-Hermitian Hamiltonian with real eigenvalues. In doing so, the well-developed machinery [9–14] of pseudo-Hermiticity has been utilized. This development, however, does not dwell upon the negativity of the *PT*-norm and invoking *C* for the positive definiteness of the *CPT*-norm. In this approach, the search for various symmetries of *H* and their identification as *C*, *PT* or *CPT* has been proposed. Despite obtaining a curiously different definition of *T* other than the simple K_0 (complex-conjugation: $K_0(AB+C) = A^*B^*+C^*$) [16], no resolution seems to have been made [18]. Also, despite this incompatibility a similar definition of the *CPT*-inner-product [16] has been adopted in [18].

The real potentials have real spectrum irrespective of their parity; they may or may not be *PT*-symmetric. The interesting unification may be brought wherein the Hamiltonians with real spectrum could be called *PT*-symmetric and *CPT*-symmetric above all. This has recently been achieved by defining the generalized parity [19] and the generalized time-reversal operator [20] and incorporating the fact that when the interaction is Hermitian, $C = P$ [16]. All Hermitian Hamiltonians have been proved to be *P*-, *T*-, *PT*-, *CPT*-invariant wherein the *CPT*-norm (*PT*-norm) is definite (indefinite) [20].

In the present work, we propose further extension of these [19, 20] definitions of *P* and *T* so as to bring consistency in proposing the *C*-, *PT*- and *CPT*-invariance of a pseudo-Hermitian Hamiltonian (real eigenvalues), the definiteness of *CPT*-norm and the indefiniteness of *PT*-norm. In our study, we prefer the use of matrix notation and matrix models of Hamiltonians. Recall that in the case of Hermiticity, for the usual stationary and time-independent states the three modifications $\Psi(x)$, $\Psi^*(x)$ and $\Psi^\dagger(x)$ usually coincide. However, in matrix notation, we have four distinct modifications of a state. These are Ψ, Ψ^* (complex-conjugate),Ψ' (transpose) and $\Psi^\dagger$ (transpose and complex-conjugate). This makes the matrix notation more general, unambiguous and unmistakable.

2. Pseudo-Hermitian matrices: a unique and definite metric

Let us note that the non-Hermitian complex matrix, *H*, given below admits real eigenvalues $E_{0,1} = a \pm \sqrt{bc}$, when $bc > 0$. We find that there exist (at least) four metrics η_i under which *H* is pseudo-Hermitian

$$H = \begin{bmatrix} a & -\mathrm{i}b \\ \mathrm{i}c & a \end{bmatrix} \qquad \eta_1 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad \eta_2 = \begin{bmatrix} r^2 & -s \\ s & 1 \end{bmatrix}$$
$$\eta_3 = \begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix} \qquad \eta_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

Here $r = \sqrt{c/b}$ and s is, in general, an arbitrary complex number, indicating that a metric need not necessarily be Hermitian. These η_1 (Pauli's σ_x) and $\eta_{2,3,4}$ have, in fact, been found by crude algebraic manipulations demonstrating that metric η is non-unique as noted earlier [10]. Furthermore, if η_1 and η_2 are found then infinitely many metrics can be constructed as

$\eta = (c_1\eta_1 + c_2\eta_2)$ provided η is invertible. On the one hand, the four metrics given above (3) do provide several operators $F_{i,j} = \eta_i \eta_j^{-1}$, $i \neq j = 1, 4$, which by commuting with H bring out its hidden symmetries [10]. In fact, the currently discussed *C*, *PT* and *CPT* symmetries shall be seen connected to $F_{i,j}$ in the examples that follow. On the other hand, the non-uniqueness of η, apart from its indefiniteness, may be undesirable as the metric determines the expectation values of various operators as $\langle\Psi|A\eta|\Psi\rangle$. We state and prove the following theorem which helps us in fixing a unique and definite metric. This could be seen as a method for finding at least one metric under which a given matrix is pseudo-Hermitian.

Theorem 1. *If a pseudo-Hermitian $n \times n$ matrix H admits real eigenvalues $(E_1, E_2, \ldots, E_n)$ and D is its diagonalizing matrix, then H is η-pseudo-Hermitian, where $\eta = (DD^\dagger)^{-1}$. The converse of this also holds.*

Proof. Let us define a real diagonal matrix $\mathcal{E} = \text{diag}[E_1, E_2, E_3, \ldots, E_n]$ such that $\mathcal{E} = \mathcal{E}^* = \mathcal{E}^\dagger$:

$$\begin{aligned} D^{-1}HD = \mathcal{E} \quad &\Rightarrow \quad D^{-1}\eta^{-1}\eta H\eta^{-1}\eta D = \mathcal{E} \quad \Rightarrow \quad (D^{-1}\eta^{-1}H^\dagger\eta D)^\dagger = \mathcal{E}^\dagger \\ &\Rightarrow \quad D^\dagger\eta^\dagger H\eta^{-1\dagger}D^{-1\dagger} = \mathcal{E}. \end{aligned} \tag{4}$$

Upon comparing the first and the last parts of the above equation, we get $D^\dagger\eta^\dagger = D^{-1}$ and $\eta^{-1\dagger}D^{-1\dagger} = D$ which are mutually consistent justifying pseudo-Hermiticity when the eigenvalues of a complex matrix are real and implying $\eta = (DD^\dagger)^{-1}$. □

In general, D will be pseudo-unitary: $D^\dagger = \delta D^{-1}\delta^{-1}$ [9, 15] with respect to some metric δ which may not be the same as η. When H is Hermitian, D will be unitary and we get $\eta = I$ as a special case.

One interesting remark is in order here: it is often not realized [8–15] that pseudo-Hermiticity does not provide a direct and explicit proof for the absolute or conditional existence of real eigenvalues. Nevertheless, it is found [8–15] to support real eigenvalues indirectly as also seen here and as follows below in the proof of the converse of the present theorem.

Proof (Converse)**.** Let

$$D^{-1}HD = \mathcal{E} \qquad \text{and} \qquad (DD^\dagger)^{-1}H(DD^\dagger) = H^\dagger \tag{5a}$$

$$\begin{aligned} &\Rightarrow \quad (D^\dagger)^{-1}(D^{-1}HD)D^\dagger = H^\dagger \quad \Rightarrow \quad (D^\dagger)^{-1}\mathcal{E}D^\dagger = H^\dagger \\ &\Rightarrow \quad D\mathcal{E}^\dagger D^{-1} = H \quad \Rightarrow \quad \mathcal{E}^\dagger = D^{-1}HD. \end{aligned} \tag{5b}$$

The first part of (5*a*) and the last part of (5*b*) imply nothing but the reality of eigenvalues. □

Similarly, when all the eigenvalues are complex-conjugate pairs and D is the diagonalizing matrix arranged such that the eigenvectors corresponding to the complex-conjugate pairs of eigenvalues remain together, then it can be proved that $\bar{\eta} = (DSD^\dagger)^{-1}$, where S is Pauli's σ_x, when H is 2×2 otherwise when H is $2n \times 2n$, S is a block-diagonal matrix: $S = \text{diag}[\sigma_x, \sigma_x, \sigma_x, \ldots, \sigma_x]$. We denote and state thus the obtained metric as

$$\eta_+ = (DD^\dagger)^{-1} \tag{6}$$

to actually see that the indefinite norm (2)

$$N_{\eta_+} = \Psi^\dagger\eta_+\Psi = \Psi^\dagger(DD^\dagger)^{-1}\Psi = \Psi^\dagger D^{\dagger^{-1}}D^{-1}\Psi = (D^{-1}\Psi)^\dagger(D^{-1}\Psi) = \chi^\dagger\chi > 0 \tag{7}$$

is now positive definite. Finding eigenvalues, eigenvectors and diagonalizing matrix is a standard exercise. In that the theorem stated and proved above is indeed an attractive proposal to find the metric for a given complex non-Hermitian matrix admitting real eigenvalues under which it is pseudo-Hermitian. However, by multiplying the columns (rows) by arbitrary constants we can get many diagonalizing matrices say D_j and this would give rise to as many metrics, say η_j, under which H will be pseudo-Hermitian. For the sake of uniqueness, one may only use η-normalized (2) eigenvectors to construct D. Earlier, it has been proved that if a pseudo-Hermitian Hamiltonian, H, has real eigenvalues then there exists an operator O such that H is pseudo-Hermitian under $(OO^\dagger)$ [10] and $(OO^\dagger)^{-1}$ [12]. Another form for η_+ in terms of the eigenvectors has also been proposed [18].

3. Construction of *C*, *P*, *T* and proposal of an inner-product

When pseudo-Hermitian Hamiltonian (1) has real eigenvalues, we have [9]

$$H\Psi_n = E_n\Psi \qquad H^\dagger\Phi_n = E_n\Phi_n \qquad \Phi_n = \eta\Psi_n. \tag{8}$$

(Ψ_n, Φ_n) are called bi-orthonormal basis. We have also witnessed in the above example (3) that several metrics could be obtained under which a given H is pseudo-Hermitian. Let us stress that this interesting practical experience remains elusive in several formal definitions. Let us examine the properties of the metrics obtained in (3). The metric η_1 is involutary ($\eta^2 = 1$). The metrics η_1, η_3, η_4 (3) are Hermitian, unitary and proper ($\det\eta = 1$). The metrics η_3, η_4 are real symmetric. The metric η_2 very importantly is non-Hermitian in general. The metrics η_1, η_4 are (constant) disentangled with the elements of H and we call them *secular* [15]. It will be very interesting to investigate whether or not one can always find an involutary and *secular* metric for an arbitrary pseudo-Hermitian matrix. The interesting exposition [10] that most of the known *PT*-symmetric Hamiltonians are actually *P*-pseudo-Hermitian is very valuable in order to connect pseudo-Hermiticity with *P* and *T* and hence to possible physical situations [15]. Once the involutary metric is found, it will be fixed for the definition of orthonormality (2) and we will take it to represent the generalized *P*. This *ad hoc* strategy also seems to have been adopted in [16]. Therefore, the question of a definition to construct *P* again, from the bi-orthonormal basis (Ψ, Φ), either does not arise or will yield $P = \eta$, eventually.

Here, one very important remark is in order: in the recent works on pseudo-Hermiticity, the indefiniteness of the η-norm (or orthonormality) has not been realized and this has given rise to an assumption that *somehow* $\Phi_n^\dagger\Psi_n$ is positive definite (e.g., equations (11) and (12) in [10], equations (5) and (6) in [12], equation (7) in [13]). Consequently, representations of **1** (the completeness) in terms of (Ψ, Φ), for instance, for the two-level matrix Hamiltonian, have been given as $\left(\Psi_0\Phi_0^\dagger + \Psi_1\Phi_1^\dagger\right)$. Though known earlier [4–9], however, the indefiniteness of the norm is centrally consequent to the novel identification of charge-conjugation symmetry by Bender *et al* [16].

Thus having fixed η, we find η-normalized (2) eigenvectors Ψ_n for H. These normalized eigenvectors are used to construct the diagonalizing matrix D and η_+ (6) which are unique only under a fixed η. We obtain another basis $\{\Upsilon_n\}$ as

$$\Upsilon_n = \eta_+\Psi_n \tag{9}$$

which, by construction (see (7)), is such that

$$\Psi_m^\dagger\Upsilon_n = \delta_{m,n}. \tag{10}$$

We propose to construct P as

$$P = \sum_{n=0}^{N}(-1)^n\Psi_n\Psi_n^\dagger \tag{11}$$

such that $P\Upsilon_n = (-1)^n\Psi_n$, implying that neither of Ψ_n, Υ_n are the eigenstates of parity as it should be. We define the anti-linear time-reversal operator T as

$$T = \left(\sum_{n=0}^{N} \Upsilon_n \Upsilon_n'\right) K_0 \tag{12}$$

such that $T\Psi_n = \Upsilon_n$ and we further have

$$PT = \left(\sum_{n=0}^{N} (-1)^n \Psi_n \Upsilon_n'\right) K_0 \tag{13}$$

such that $PT\Psi_n = (-1)^n\Psi_n$. We adopt the definition of C as proposed in [18]

$$C = \sum_{n=0}^{N} (-1)^n \Psi_n \Upsilon_n^\dagger \qquad \text{where} \qquad \sum_{n=0}^{N} \Psi_n \Upsilon_n^\dagger = 1 \tag{14}$$

such that $C\Psi_n = (-1)^n\Psi_n$. Next using (13) and (14) the symmetry operator CPT takes the form

$$CPT = \left(\sum_{n=0}^{N} \Psi_n \Upsilon_n'\right) K_0 \tag{15}$$

such that $CPT\Psi_n = \Psi_n$ The following involutions,

$$(CPT)^2 = (PT)^2 = C^2 = 1 \tag{16}$$

always hold. However, we get

$$T^2 = P^2 \qquad \text{iff} \quad (-1)^{m+n}\Psi_m^\dagger\Psi_n = \Upsilon_m^\dagger\Upsilon_n. \tag{17}$$

When the Hamiltonian is Hermitian, P and T have been proved to be involutary [20]. However, for a pseudo-Hermitian Hamiltonian this becomes conditional. In equation (87) of [18], the above condition is suggested to ensure that P and T are involutary. Let us remark that this condition only ensures that $P^2 = T^2$. Further, since we choose P to be involutary, so will T be. We find the following commutation relations,

$$[H, C] = [H, PT] = [H, CPT] = 0 \qquad \text{and} \qquad [H, P] \neq 0 \neq [H, T] \tag{18}$$

displaying the invariance and non-invariance of the Hamiltonian. We now define an X-inner-product as

$$\langle\cdot|\cdot\rangle = (X\Psi_m)^\dagger\Upsilon_n = (X\Psi_m)^\dagger\eta_+\Psi_n = \epsilon_n\delta_{m,n}. \tag{19}$$

and consequently, the X-norm as

$$N_{X,n} = (X\Psi_n)^\dagger\Upsilon_n = (X\Psi_n)^\dagger\eta_+\Psi_n. \tag{20}$$

Here X represents the operators corresponding to discrete symmetries: C, PT and CPT as constructed above, such that $[H, X] = 0$. Since $X\Psi_n = \epsilon_n\Psi$, ϵ_n is real, the X-inner-product in view of (7) will be real definite.

The definition of T and the inner-product in [16, equations (5), (12) and (22)] in our notation read

$$T = K_0 \qquad \langle\cdot|\cdot\rangle = (X\Psi_m)'\Psi_n \tag{21}$$

which is not real definite in general, noting the fact that Ψ_n are eigenvectors over a complex field (the elements of these vectors are complex).

The definition of T and the inner-product proposed in [18, equations (78) and (75)] would read

$$T = \left(\sum_{n=0}(-1)^n \Upsilon_n \Upsilon_n'\right) K_0 \qquad \langle\cdot|\cdot\rangle = (X\Upsilon_m)'\Psi_n. \tag{22}$$

Once again the inner-product is not real definite. We have earlier [20] proved and illustrated that the definition of the inner-product (21) [16] does not let the energy eigenstates of the Hermitian H be orthogonal. We would like to claim that our definition of the X-inner-product proposed here is the most general and consistent so far [3–9, 16, 18–20], for the PT-symmetric or pseudo-Hermitian Hamiltonians.

Let us remark that the inner-product (21) works in [16] since the Hamiltonian considered there is complex symmetric ($H' = H$) which is adjudged to be pseudo-Hermitian under $\eta = \sigma_x$, here. Thus the eigenvectors are orthogonal as $\psi_0'\psi_1 = 0$, in addition to their η-orthogonality: $\psi_0^\dagger \eta \psi_1 = 0$. For the case when

$$H = \begin{bmatrix} a-c & \mathrm{i}b \\ \mathrm{i}b & a+c \end{bmatrix} \qquad \eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P \qquad \psi_0 = \begin{bmatrix} 1 \\ -\mathrm{i}r \end{bmatrix}$$
$$\psi_1 = \begin{bmatrix} 1 \\ -\mathrm{i}/r \end{bmatrix} \qquad r = \frac{a \pm \sqrt{c^2-b^2}}{b} \tag{23}$$

the prescription of [16] would again work. It will, however, fail for the models in the illustrations, I_1 and I_2, given below.

4. Illustrations

In the definitions for the construction of P, T, C, though general, certain features can still not be proved. For instance whether C and P will always not commute and whether P and T will always commute. Given that P for a Hamiltonian is non-involutary will we get an involutary T? Can we get C to be *secular*? In this regard, simple doable examples are desirable. In the following we present two illustrations to throw some more light on the unanswered questions stated here.

Without loss of generality, we take 2×2 matrix Hamiltonians [15] and construct P, T, C as per equations (11), (12) and (14) as

$$P = \Psi_0\Psi_0^\dagger - \Psi_1\Psi_1^\dagger \qquad T = (\Upsilon_0\Upsilon_0' + \Upsilon_1\Upsilon_1')K_0 \qquad C = \Psi_0\Upsilon_0^\dagger - \Psi_1\Upsilon_1^\dagger \tag{24}$$

for short. In illustration I_1, we take up the same Hamiltonian as given in (3). Here the fundamental metric (P) is involutary. In illustration I_2, it is kept non-involutary.

I_1. We take the pseudo-Hermitian Hamiltonian, H, and the fundamental metric, $\eta\,(=\eta_1)$, from (3). The η-normalized eigenvectors are

$$\Psi_0 = \sqrt{\frac{r}{2}}\begin{bmatrix} -\mathrm{i}/r \\ 1 \end{bmatrix} \qquad \Psi_1 = \sqrt{\frac{r}{2}}\begin{bmatrix} 1/r \\ -\mathrm{i} \end{bmatrix}. \tag{25}$$

One can readily check that $\Psi_0^\dagger\eta\Psi_1 = 0$, but $\Psi_0'\Psi_1 = -\mathrm{i}\frac{1+r^2}{2r} \neq 0$ for the approach [16] to work here. Following section 4 , we obtain P, T and η_+ as

$$P = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix} K_0 \qquad \eta_+ = \begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix} \tag{26}$$

where r is essentially real which is unlike the unrestricted r in (3). Note that P turns out to be the same as η_1—the chosen fundamental metric. The symmetry operators C, PT and CPT are

$$C = \begin{bmatrix} 0 & -\mathrm{i}/r \\ \mathrm{i}r & 0 \end{bmatrix} \qquad PT = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} K_0 \qquad CPT = \begin{bmatrix} 0 & -\mathrm{i}/r \\ -\mathrm{i}r & 0 \end{bmatrix} K_0. \tag{27}$$

One can readily check that the orthogonality conditions arising from (21) and (22) do not work here, e.g. $(CPT\Psi_0)'\Psi_1 \neq 0 \neq (PT\Psi_0)'\Psi_1$. However, the proposed orthogonality arising from (20) works. The symmetry C could be checked to be identical to $\eta_1\eta_3^{-1}$ (see (3)), demonstrating how two distinct metrics combine to yield a hidden symmetry of the Hamiltonian. In addition to the general results stated above, we get $(CP)^{-1} = PC = \eta_+$; in actual $\mathcal{CPT}$-invariance $\mathcal{C}$, $\mathcal{P}$ do commute [17]. We also confirm the commutation of P and T and the involutions: $T^2 = P^2 = 1 = C^2$. Here P and T are *secular* whereas C turns out to depend on the elements of H. Similar experience can be had by studying the model of [16] and of (23). Interestingly, the fundamental metrics in these cases are Pauli's matrices which are involutary, Hermitian, unitary, proper and also *secular*.

I_2. In the following, let us now take an example where the fundamental metric is only Hermitian and *secular* as it does not affect the eigenvalues: $E_{0,1} = \frac{1}{2}\left[(a+b) \pm \sqrt{(a-b)^2 + 4c^2}\right]$. We introduce $\theta = \frac{1}{2}\tan^{-1}\frac{2c}{a-b}$.

$$\begin{aligned} H &= \begin{bmatrix} a & -\mathrm{i}c/x \\ \mathrm{i}cx & b \end{bmatrix} \qquad \eta = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \\ \Psi_0 &= \sqrt{x}\begin{bmatrix} \cos\theta/x \\ \mathrm{i}\sin\theta \end{bmatrix} \qquad \Psi_1 = \frac{1}{\sqrt{x}}\begin{bmatrix} \mathrm{i}\sin\theta \\ x\cos\theta \end{bmatrix}. \end{aligned} \tag{28}$$

Check that the states are only η-orthogonal and we have $\Psi_0'\Psi_1 = \mathrm{i}\sin 2\theta(1+x^2)/(2x) \neq 0$ as in I_1, indicating once again the failure of the inner-products (21) and (22). We construct P, T, C as

$$\begin{aligned} P &= \begin{bmatrix} \frac{\cos 2\theta}{x} & -\mathrm{i}\sin 2\theta \\ \mathrm{i}\sin 2\theta & -x\cos 2\theta \end{bmatrix} \qquad T = \begin{bmatrix} x\cos 2\theta & \mathrm{i}\sin 2\theta \\ \mathrm{i}\sin 2\theta & \frac{\cos 2\theta}{x} \end{bmatrix} K_0 \\ C &= \begin{bmatrix} \cos 2\theta & -\frac{\mathrm{i}\sin 2\theta}{x} \\ \mathrm{i}x\sin 2\theta & -\cos 2\theta \end{bmatrix} \end{aligned} \tag{29}$$

and η is returned as η_+. Very interestingly, P is different from the fundamental metric η. Since this fundamental metric is definite giving $\Psi_n^\dagger\eta\Psi_n = +1$, the construction of η_+ as per (6) yields it back. Unlike other examples here we have $T^2 \neq P^2 \neq 1$, whereas the results (16) are met here. We find that P and T commute; C and P do not commute. We get $PC \neq (CP)^{-1} = \eta_+ = \eta$. When $x = 1$, the scenario for Hermiticity can be observed. In this model all C, P and T are *non-secular* as they depend on the elements of the Hamiltonian.

5. Conclusions

The theorem stated and proved in section 3 adds an important result in matrix algebra [9] for constructing a metric(s) $\eta_+ = (DD^\dagger)^{-1}$ (6) where D is the diagonalizing matrix for the pseudo-Hermitian matrix which has real eigenvalues. The proven positive definiteness (7) of this metric is of utility while constructing the generalized P, T, C and an inner-product for a matrix Hamiltonian which possesses a real spectrum.

If X is a discrete symmetry operator for the Hamiltonian H, i.e. $[X, H] = 0$, then the proposed definition of the inner-product, $\langle\cdot|\cdot\rangle$, of the eigenstates of H as $\langle X\Psi|\eta_+\Psi\rangle$ (19) or even $\langle X\Psi|\eta\Psi\rangle$ is the most general definition proposed so far when the Hamiltonians are *PT*-symmetric or η-pseudo-Hermitian.

Our construction of operators C, P and T for pseudo-Hermitian Hamiltonians (with real spectrum) is essentially compatible with the indefiniteness of the *PT*-norm and the definiteness of the *CPT*-norm. It could now be asserted that Hamiltonians with real eigenvalues are *CPT*-invariant and the *CPT*-norm is positive definite. The models considered here are matrix Hamiltonians, however, other types of Hamiltonians are still desired to be included.

One point that requires emphasis is: in pseudo-Hermiticity, we are able to construct only *three* distinct involutary operators, which we have designated as P, T and C analogous to the conventional $\mathcal{P}$, $\mathcal{T}$ and C [17]. Admittedly, the only properties possessed by P, T and C are their involutions (16), various commutations and non-commutations (18), and the inner-product (19), to strike their correspondence with the actual $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{C}$ of the Hermitian field theory [17]. Much deeper connections and arguments would be required to make claims in the style of the conventional $\mathcal{CPT}$-invariance. Our matrix Hamiltonians could be taken as toy models of a future pseudo-Hermitian field theory.

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